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FINITE STATE BILATERALLY DETERMINISTIC STRONGLY MIXING PROCESSES

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ABSTRACT

Triviality of the two-sided tail field of a stationary process is not an invariant property under factorization ([4]). In this paper we give an example of a bilaterally deterministic process with finitely many states which is strongly mixing. This extends and complements a result of Bradley ([1]).

1. Introduction

Let $\mathbf{X} = \{X_n\}_{n=-\infty}^{\infty}$ be a finite state stationary process. A factor process of $\mathbf{X}$, $f(X)$, is a stationary process $\mathbf{Y} = \{Y_n\}_{n=-\infty}^{\infty}$ which is obtained as follows: Let f_0: $\mathbf{X} \to$ finite set, f_0 measurable, $Y_0 = f_0(\mathbf{X})$ and $Y_n = f_0(\sigma^n\, \mathbf{X})$ where σ

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is the shift, $(\sigma\mathbf{X})_i = X_{i+1}$. $\mathbf{Y}$ is an isomorphic factor if f may be taken to be invertible.

A property of a stationary process is an invariant property if it is also shared by all its factor processes. For example, the Kolmogorov zero-one law is an invariant property. This law says that the tail σ-algebra is trivial. A stronger property is that the two-sided tail is trivial. This σ-algebra is defined as follows: Let $\mathcal{F}_n^+ = \sigma(X_n, X_{n+1}, \ldots)$ be the σ-algebra generated by $X_n, X_{n+1}, \ldots$ and $\mathcal{F}_n^- = \sigma(X_{-n}, X_{-n-1}, \ldots)$ be the σ-algebra generated by $X_{-n}, X_{-n-1}, \ldots$. Then, the two-sided tail of $\mathbf{X}$ is defined by

$$\bigcap_{n=1}^{\infty} \mathcal{F}_n^+ \vee \mathcal{F}_n^-.$$

If $\mathbf{X}$ is an independent, identically distributed (i.i.d.) process, then the two-sided tail is trivial. But, it was shown by Ornstein and Weiss ([4]) that this is not an invariant property, and that any stationary process has a factor (actually isomorphic) process such that the two-sided tail is (up to sets of measure zero) the total σ-algebra generated by the process. Such a process is called **bilaterally deterministic**.

A mixing property, which Ornstein showed to be invariant, is the very weak Bernoulli property (VWB). However, there are a number of mixing properties known in probability theory which are non-invariant. One example is α-mixing (strong mixing or Rosenblatt mixing ([5])) which says that given $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that, if $n \geq n_0(\varepsilon)$,

$$\sup_{A \in \mathcal{F}_n^+;\, B \in \mathcal{F}_n^-} |P(A \cap B) - P(A)P(B)| \leq \varepsilon.$$

It was shown in [6] that there is a VWB process which is not ϕ-mixing (see [2] for a definition and the relation to strong mixing). In fact, VWB processes need not to be α-mixing as a variant on the examples given in [4] show. On the other hand α-mixing does imply the Kolmogorov 0-1 law.

A stronger mixing condition than α-mixing is ρ-mixing, which is the following (see [2]): Given $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for $n \geq n_0(\varepsilon)$

$$\sup\,\{\rho(g,h)\colon g \in \mathcal{L}^2(X) \text{ is } \mathcal{F}_n^+\text{-measurable, } h \in \mathcal{L}^2(\mathbf{X}) \text{ is } \mathcal{F}_n^-\text{-measurable}\} < \varepsilon,$$

where $\rho(g,h)$ is the correlation coefficient. Bradley ([1]) constructed a process which is bilaterally deterministic and yet is ρ-mixing. However, the constructed process has the disadvantage of being a continuous state process. He left it as an open problem to construct a finite state process with such properties. We give a partial answer to this problem by constructing a finite state bilaterally deterministic process which is α-mixing. It is not clear that pushing our method further will yield a ρ-mixing process as well.

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2. A construction of bilaterally deterministic finite state processes

Let $\mathbf{X} = (X_n;\ n \in \mathbb{Z})$ be a sequence of independent, identically distributed random variables with common distribution function $P(X_1 = k) = \frac{1}{3}$ for $k = 0, 1, 2$. In this section we shall describe a construction method to define a new process $(Z_n;\ n \in \mathbb{Z})$ as a factor of the independent process $\mathbf{X}$, the state space being $E = \{0,\ 1\ ,2,\ (2,0),\ (2,1),\ (2,2)\}$. In special cases, this factor process turns out to be bilaterally deterministic and strongly mixing (Section 3).

The three basic ideas in our construction are as follows:

We first introduce markers for $\mathbf{X}$ as a finite union of cylinder sets each containing subblocks of 2's of certain lengths $N \in \mathbf{J}$, $\mathbf{J}$ a fixed subset of $\mathbb{N}$.

Then, in between certain pairs of successive markers associated to N, we code the parity vector of points which are marked in this way. The components of the parity vector are the mod three sums of the coordinates (in between the two markers) at locations whose differences are a multiple of $K(N)$ where $K(N)$ is some fixed odd integer. The parity vector allows to rewrite subblocks of length $K(N)$ within the markers, once all other coordinates in between the markers are known. This will yield a bilaterally deterministic stationary process, provided $\mathbf{J}$ is thin enough.

In order to achieve the strong mixing property, we need to create enough randomness in the construction (equivalently, not to lose too much of randomness of the i.i.d. process $\mathbf{X}$). Let two consecutive markers associated to N be fixed.

The coordinates of length log (N) following the subblock of 2's of length N determines a (random) subblock of length $K(N)$ in the second marker and this subblock will be added (mod 3) to the parity vector. This means that the subblock of length $K(N)$ thus obtained still has the information needed to recover a subblock of length $K(N)$ in the original sequence, but also becomes in a certain way 'random'. To make this more precise observe that we cannot expect stronger mixing properties than ρ-mixing (i.e. φ- or ψ-mixing (see [2])). Thus the measures of intersections of cylinders in the past and future cannot be uniformly approximated by the corresponding product measure. In fact, it will turn out that these intersections are either empty or have measure equal to $3^{K(N)}$ times the product measure. Moreover, for a cylinder in the past, the number of nonempty intersections with cylinders in the future is approximately $3^{-K(N)}$ times the total number of such sets.

We now start with a detailed description of our construction.

Let K and N be positive integers with $K \leq N$ and $\log N$ an integer. Here and elsewhere in this paper we take logarithms to base 3. The (K, N)-marker at time $t \in \mathbb{Z}$ is the event $E_t\,(K, N)$ defined by

$$\begin{aligned} X_l &= 2 \quad \text{if } l = t-1,\ t-2, \ldots,\ t-N, \\ X_l &\neq 2 \quad \text{if } l = t,\ t+1+\log\ N, \\ &\qquad\qquad t-N-1-j\,(K+1) \quad \text{for } j = 0, 1, \ldots,\ N. \end{aligned}$$

Clearly $P[E_t\,(K, N)] = 2^{N+3}\,3^{-2N-3}$.

We define some notation to describe (K, N)-markers, suppressing the dependence on (K, N). This dependence will be clear from the context when we use it.

The smallest coordinate restricted by the marker we write as $\alpha(t) = t - N - 1 - N\ (K+1)$, the largest coordinate of the marker is $\omega(t) = t + 1 + \log\ N$. The length of the marker is denoted by $\Lambda = \omega(t) - \alpha(t) + 1 = N(K+2) + \log\ N + 3$. Also set

$$\begin{aligned} I(t) = \{u\colon t - N - 1 - j(K+1) < u < t - N - 1 - (j-1)(K+1) \\ \text{for some } \ j,\ 1 \leq j \leq N\} \end{aligned}$$

which are the free coordinates in the beginning of the marker. Similarly the free coordinates at the end of the marker are

$$J(t) = \{u\colon t < u < t + \log\ N + 1\}\,.$$

Now fix a strictly increasing sequence of positive integers $\{N_i;\ i \geq 1\}$ with $\log\ N_i$ an integer and fix integers $K_i \leq N_i$, $i \geq 1$. Write $\alpha_i(t)$, $\omega_i(t)$, Λ_i, $J_i(t)$ and $I_i(t)$ accordingly for (K_i, N_i)-markers at time t. Define the waiting times T_i (for the process $\mathbf{X}$) by

$$T_i = \inf\ \{t\colon \alpha_i(t) \geq 0,\ E_t\ (K_i,\ N_i) \text{ occurs}\}.$$

Note that $P(T_i < \infty \text{ for every } i \geq 1) = 1$ and even $E\,T_i < \infty$ for each $i \geq 1$. This is in the proof of our first lemma.

LEMMA 1: *For $\beta, \varepsilon > 0$ there is an i_0 such that for each $i \geq i_0$ there is a γ satisfying*

$$P(\beta < T_i < \gamma) \geq 1 - \varepsilon.$$

Proof: In order to see that $E\,T_i$ is finite, first note that the length Λ_i of a (K_i, N_i)-marker is less than $2N_i^2$, because $K_i \leq N_i$. Next, define $\tilde{T}_i = \inf\ \{t \geq 1\colon\ \alpha_i(2tN_i^2) \geq 0\ \ E_{2N_i^2 t}(K_i, N_i) \text{ occurs}\}$. Note that $\tilde{T}_i < \infty$ a.s. and that $\tilde{T}_i$ is a geometric random variable, since the events $E_{2N_i^2 t}(K_i,\ N_i)$ for $t = 1, 2, 3, \ldots$ are independent. Therefore

$$T_i \leq 2N_i^2\ \tilde{T}_i$$

and $E\,T_i < \infty$.

Choose i_0 so large that for $i \geq i_0$

$$\beta 2^{N_i+3}\ 3^{-2N_i-3} < \varepsilon/2$$

and then choose γ so large, depending on i, that

$$E\,T_i < \varepsilon\gamma/2.$$

Then

$$\begin{aligned} P(T_i \leq \beta \quad \text{or} \quad \gamma \leq T_i) &\leq P(T_i \leq \beta) + P(T_i \geq \gamma) \\ &\leq \beta P(E_{2N_i^2}(K_i, N_i)) + \frac{E\,T_i}{\gamma} \leq \varepsilon. \qquad \blacksquare \end{aligned}$$

By the above lemma, we can find an increasing sequence $\mathbf{J} = \{N_i\colon i \geq 1\}$, $\log\ N_i \in \mathbb{N}$, and positive integers $\beta_i < \gamma_i$, such that for any choice of $K_i \leq \log\ N_i$, $K_i \uparrow \infty$,

$$P(\beta_i < \alpha_i(T_i) < \omega_i(T_i) < \gamma_i) > 1 - 2^{-i-4}$$

and that

$$\beta_i + \Lambda_i < \gamma_i < \beta_{i+1} - \Lambda_{i+1} \quad \text{and} \quad \Lambda_{i+1} > 3\Lambda_i + 2\gamma_i.$$

We further assume that the sequence N_i grows fast enough that

$$\sum_{n \geq N_i} n^2\, 3^{-n/2} < 2^{-i-3}.$$

Now we define the process $\mathbf{Z} = \{Z_n : n \in \mathbb{Z}\}$.

We will set $Z_n = X_n$ unless for some $i \geq 1$ all of the following occurs:

(1) $E_t\ (K_i,\ N_i)$ for some t with $t - K_i \leq n \leq t - 1$;

(2) $E_s\ (K_i,\ N_i)$ for some $s < t$ with $2\beta_i < t - s < 2\gamma_i$ and for all r, $s < r < t$, $E_r(K_i,\ N_i)$ does not occur; and

(3) if $E_r(K_{i'},\ N_{i'})$ occurs for $i' > i$, then either $r + \Lambda_{i'} < \alpha_i(s)$ or $r - \Lambda_{i'} > \omega_i(t)$.

If conditions (1), (2) and (3) are satisfied we set $Z_n = (2, p)$, where p is defined as follows.

Define $l = t - n$. We say the choice determined by the marker at s is

$$q = q(x, s) = \sum_{u=1}^{\log N_i} X_{s+u}\, 3^{u-1}$$

so $0 \leq q < N_i$. We set

$$Z_n = (2, p)$$

where

$$p = X_{\alpha_i(t)+q(K_i+1)+l} + \sum_{j:\ 0 < jK_i < \alpha_i(t) - l - \omega_i(s) - K_i} X_{\omega_i(s)+l+jK_i} \pmod 3.$$

Note that if j' is fixed as in the sum and $u = \omega_i(s) + l + jK_i$, then X_u is determined by Z_n and the other values of X_r appearing in the definition of p. In particular, X_u is determined by the values of $\{Z_r;\ |r - u| \geq K_i\}$.

Formally, let

$$\begin{aligned} F_n = \bigcup_{i \geq 1} \bigcup_{t - K_i \leq n \leq t-1} \Big\{ E_t(K_i, N_i) \cap \\ \bigcup_{2\beta_i < t - s < 2\gamma_i} \Big(\Big[E_s(K_i, N_i) \cap E_{s+1}(K_i, N_i)^c \cap \\ \cap \cdots \cap E_{t-1}(K_i, N_i)^c \Big] \cap \bigcap_{i' > i} \bigcap_{r = \alpha_i(s) - \Lambda_{i'}}^{\omega_i(t) + \Lambda_{i'}} E_r(K_{i'}, N_{i'})^c \Big) \Big\}; \end{aligned} \tag{2.1}$$

then

$$Z_n = (X_n, p)1_{F_n} + X_n(1 - 1_{F_n}).$$

Let $-\infty \le a \le b \le \infty$ and define

$$\mathcal{F}_a^b = \sigma(X_u;\ a \le u \le b) \quad \text{and} \quad \mathcal{G}_a^b = \sigma(Z_u;\ a \le u \le b).$$

We have that $\mathcal{F}_a^b \subset \mathcal{G}_a^b$ for all a, b and, by construction, $\mathcal{F}_{-\infty}^{\infty} = \mathcal{G}_{-\infty}^{\infty}$.

Define

$$t_i = \inf\{t \ge 0\colon\ E_t(K_i, N_i) \text{ occurs}\} \quad \text{and} \quad s_i = \sup\{s < 0\colon\ E_s(K_i, N_i) \text{ occurs}\}.$$

Note that $t_i \le T_i$ (defined before) for all $i \ge 1$.

For $l > 1$ define C_l to be the event such that

(i) $\forall i \ge l$

$$\beta_i < -\omega_i(s_i) < -\alpha_i(s_i) < \gamma_i \quad \text{and} \quad \beta_i < \alpha_i(t_i) < \omega_i(t_i) < \gamma_i;$$

and

(ii) $\forall i \ge l$, $\exists$ no string of consecutive $N_i/2$ occurrences of "2" in the $\{X_n\}$ process within Λ_i of 0 to the right and also no such string of "2"s to the left of 0.

Observe that on C_l for each $i \ge l$

$$Z_{t_i - 1} \notin \{0, 1, 2\},$$

i.e. the **Z**-process differs from the **X**-process.

A **mixed cylinder** is an event of the form

$$A = \{X_u = a_u, Z_v = b_v \text{ for } u \in I,\ v \in J\}$$

where $I, J \subseteq \mathbb{Z}$ are finite and disjoint and

$$a_u \in \{0, 1, 2\}, \quad b_v \in \{0, 1, 2, (2,0), (2,1), (2,2)\}.$$

If $J = \emptyset$, A is an **X**-cylinder and, if $I = \emptyset$, A is a **Z**-cylinder.

If A is a mixed cylinder and $K \subseteq \mathbb{Z}$, denote $\pi_K(A)$ to be the projection of A onto the K-coordinates, that is

$$\pi_K(A) = \{X_u = a_u;\ Z_v = b_v \text{ for } u \in I \cap K,\ v \in J \cap K\}.$$

Define $\mathcal{H}_l$ to be the σ-algebra generated by all mixed cylinders A with coordinate set $[0, \omega_{i_0}(t_{i_0})]$ for some $i_0 \geq l$ and $\mathbf{X}$-coordinates for all u,

$$\omega_i(t_i) < u < \alpha_{i+1}(t_{i+1}) \quad \forall l \leq i \leq i_0,$$
$$0 \leq u < \alpha_l(t_l)$$

and $A \cap C_l \neq \emptyset$. Let $E'_l =$ the event that $E_t(K_l,\ N_l)$ occurs for the t for which $\alpha_l(t) = 0$. Set

$$D_l = \bigcap_{i>l}\{t_i > N_i^2\} \cap \bigcap_{i \geq l}\{s_i \leq -N_i\}.$$

We have

LEMMA 2:
(1) $\mathcal{G}_{-\infty}^{-\beta_l} \cap C_l \subseteq \mathcal{F}_{-\infty}^{-\beta_l+\Lambda_l} \cap C_l$,
(2) $\mathcal{G}_{\Lambda_l}^{\infty} \cap C_l \subseteq \mathcal{H}_l \cap C_l$,
(3) $\mathcal{G}_{-\infty}^{-\Lambda_l-1} \cap E'_l \cap D_l = \mathcal{F}_{-\infty}^{-1} \cap E'_l \cap D_l$.

Proof: (1) We must show that for $j \leq -\beta_l$, Z_j conditioned on C_l is $\mathcal{F}_{-\infty}^{-\beta_l+\Lambda_l} \cap C_l$ measurable. Fix $j \leq -\beta_l$.

Let $F = F_j$ where F_j has been defined in (2.1). Then

$$Z_j = (X_j,\ \nu)\, 1_F + X_j(1 - 1_F)$$

where ν is, by definition, $\mathcal{F}^j_{-\infty}$-measurable. Thus it suffices to show that

$$F \cap C_l \in \mathcal{F}_{-\infty}^{-\beta_l+\Lambda_l} \cap C_l$$

and that

$$F^c \cap C_l \in \mathcal{F}_{-\infty}^{-\beta_l+\Lambda_l} \cap C_l.$$

For $i \geq 1$ and t with $1 \leq t - j \leq K_i$ one has:
(i) If $i < l$,

$$E_t(K_i, N_i)^c, E_t(K_i, N_i) \in \mathcal{F}_{\alpha_i(t)}^{\omega_i(t)} \subset \mathcal{F}_{-\infty}^{j+\Lambda_i} \subset \mathcal{F}_{-\infty}^{-\beta_l+\Lambda_l}.$$

(ii) If $i \geq l$, by definition of C_l and since $t \leq -\beta_l + K_i < 0$,

$$E_t(K_i, N_i)^c \cap C_l,\ E_t(K_i, N_i) \cap C_l \in \mathcal{F}_{-\infty}^{-\beta_l} \cap C_l.$$

Using the definition of C_l it is straightforward to see that

$$F \cap C_l = \bigcup_{i \geq 1} \bigcup_{t-K_i \leq j \leq t-1} \Bigg\{ E_t(K_i, N_i)$$

$$\cap \bigcup_{2\beta_i < t-s < 2\gamma_i} \bigg(\Big[E_s(K_i, N_i) \cap E_{s+1}(K_i, N_i)^c \cap \cdots \cap E_{t-1}(K_i, N_i)^c \Big]$$

$$\cap \bigcap_{i' > \max(i,l)} \bigcap_{r=\alpha_i(s)-\Lambda_{i'}}^{-\beta_l-\Lambda_{i'}} E_r(K_{i'}, N_{i'})^c \cap \bigcap_{i<i'<l} \bigcap_{r=\alpha_i(s)-\Lambda_{i'}}^{\omega_i(t)+\Lambda_{i'}} E_r(K_{i'}, N_{i'})^c \bigg) \Bigg\} \cap C_l,$$

and by (i) and (ii), (1) follows.

(2), (3) The proofs are similar to (1) and left to the reader.

LEMMA 3: $P(C_l) \geq 1 - 2^{-l}$.

Proof: For a point in C_l^c, there is a string of "2" of length $N_i/2$ for $i \geq l$ within Λ_i of the origin or the closest markers to zero are not in the appropriate intervals defined by β_i and γ_i. Therefore, by the choice of β_i and γ_i, and since $T_i \leq t_i$,

$$\begin{aligned} P(C_l^c) &\leq 2 \sum_{i \geq l} 2^{-i-4} + 2 \sum_{i \geq l} \Lambda_i \, 3^{-N_i/2} \\ &\leq 2^{-l-2} + 4 \sum_{i \geq l} N_i^2 \, 3^{-N_i/2} \leq 2^{-l}. \qquad \blacksquare \end{aligned}$$

THEOREM 1: *$\{Z_n\}_{n=-\infty}^{\infty}$ as constructed above is bilaterally deterministic.*

Proof: We must show $\mathcal{G}_{-\infty}^{-n} \vee \mathcal{G}_n^{\infty} = \mathcal{G}_{-\infty}^{\infty}$. Since $\mathcal{G}_{-\infty}^{\infty} = \mathcal{F}_{-\infty}^{\infty}$, it is enough to show that every $\mathcal{F}_{-n+1}^{n-1}$ cylinder is in $\mathcal{G}_{-\infty}^{-n} \vee \mathcal{G}_n^{\infty}$. To this end it is enough to show that every cylinder of the form $A = \{X_u = a\}$ for $a \in \{0,1,2\}$ and $-n < u < n$ belongs to $\mathcal{G}_{-\infty}^{-n} \vee \mathcal{G}_n^{\infty}$. Let $x \in A$. Since $P\left(\bigcup_{l=1}^{\infty} C_l\right) = 1$ there is an l so $x \in C_l$ with probability one. Find $l' \geq l$ so $K_{l'} > 2n$. Then by our construction X_u is determined by finitely many coordinates of

$$\{Z_v \colon |v-u| > 2n\}.$$

Let the corresponding cylinder set in $\mathcal{G}_{-\infty}^{-n} \vee \mathcal{G}_n^{\infty}$ be denoted by $A(x)$. Thus a.s. A is a countable union

$$A = \bigcup_x A(x)$$

and belongs to $\mathcal{G}_{-\infty}^{-n} \vee \mathcal{G}_n^{\infty}$. $\blacksquare$

3. The strong mixing property of $\{Z_n\}_{n\in\mathbb{Z}}$

In this section we prove the strong mixing property of the process $\{Z_n\}_{n\in\mathbb{Z}}$ constructed in Section 2.

THEOREM 2: *Let $\delta_i > 0$ so that $\sum_{i=1}^{\infty} \delta_i < \infty$. Under the assumptions in Section 2, and if*

$$\sum_{i=1}^{\infty} \frac{3^{K_i}}{\delta_i^2\, N_i} < \infty,$$

the process $\mathbf{Z} = \{Z_n; n \in \mathbb{Z}\}$ is bilaterally deterministic and strongly mixing.

In the remaining part we shall prove this statement. To get an early idea of what has to be proved and how to reduce the problem, the reader may consult the proof of Theorem 2 at the end. We begin with the definition of frames.

Definition: Let $l < L$ be integers.

A **frame** is an $\mathbf{X}$-cylinder $S = \{X_u = a_u \colon u \in I\}$ with times $s_L < s_{L-1} < \cdots < s_l < 0 \le t_l < t_{l+1} < \cdots < t_L$ so that:

(1) $S \subseteq E_{t_i}(K_i,\, N_i) \cap E_{s_i}(K_i,\, N_i)$ $(l \le i \le L)$ and if $s_i < r < t_i$ then $S \cap E_r(K_i,\, N_i) = \emptyset$.

(2) $I = \{u \in [\alpha_L(s_L), \omega_L(t_L)] \colon u \notin J_i(s_i), u \notin I_i(t_i), l \le i \le L\}$.

(3) $S \cap C_l \neq \emptyset$.

We denote the forward frame of S

$$S^+ = \{X_u = a_u \colon u \in I \cap [0, \infty)\}$$

and the backward frame of S

$$S^- = \{X_u = a_u \colon u \in I \cap (-\infty, 0)\}.$$

Note that if $S(1)$, $S(2)$ are frames with times

$$s_L(k) < s_{L-1}(k) < \cdots \le s_l(k) < 0 \le t_l(k) < \cdots < t_L(k) \qquad (k = 1, 2),$$

then $S(1)^- \cap S(2)^+$ is also a frame with times

$$s_L(1) < s_{L-1}(1) < \cdots < s_l(1) < 0 \le t_l(2) < \cdots < t_L(2).$$

Also, by independence, we have for any frame

$$P(S) = P(S^-)\, P(S^+).$$

In the discussion that follows let $l < L$ be fixed positive integers.

Definition: Let S^+ be a fixed forward frame (with indices $l < L$) and let $l \leq i_0 \leq L$. Define $\mathcal{L}(i_0)$ to be the set of all step functions $f = \sum f_A 1_A$ such that

(i) $0 \leq f_A \leq 1$ and A has coordinate set $[0, \omega_L(t_L)]$,

(ii) A is a mixed cylinder, $A \subseteq S^+$,

(iii) A has $\mathbf{Z}$-coordinates for $u \in \bigcup_{i=l}^{i_0}(t_i - K_i - 1,\ t_i)$, and

(iv) A has all other coordinates in $[0, \omega_L(t_L)]$ as $\mathbf{X}$-coordinates.

Note that the representation $f = \sum f_A 1_A$ is unique (if each A appears at most once).

As a convention we set

$$\mathcal{L}(l-1) = \{f = \sum f_A 1_A\colon A \text{ is an } \mathbf{X}\text{-cylinder with } [0, \omega_L(t_L)] \text{ as coordinate set and } A \subseteq S^+\}.$$

For the next part of the discussion we fix a frame S with times $s_L < s_{L-1} < \cdots < s_l < 0 \leq t_l < \cdots < t_{L-1} < t_L$. Let B_0 be a (disjoint) collection of $\mathbf{X}$-cylinders with coordinate set $[\alpha_L(s_L), -1]$ and contained in S^-.

Let $l \leq i_0 \leq L$ and let $f \in \mathcal{L}(i_0)$ with canonical representation $f = \sum f_A 1_A$.

Let B be one of the cylinders in B_0 and A one of the cylinders in the canonical representation of f. Recall that π_K denotes 'projection' onto the K-coordinates.

Set

$$\overline{B} = \pi_{J_{i_0}(s_{i_0})^c}(B) \cap \pi_{[0,\alpha_{i_0}(t_{i_0})-1]}(A), \quad \overline{A} = \pi_{I_{i_0}(t_{i_0})^c} \circ \pi_{[\alpha_{i_0}(t_{i_0}),\infty)}(A)$$

and

$$\tilde{A} = \left[\pi_{[t_{i_0}-N_{i_0}, t_{i_0}-1]^c}(\overline{A})\right] \cap \{X_u = 2\colon t_{i_0} - N_{i_0} \leq u \leq t_{i_0} - 1\}.$$

If

$$\pi_{[0,\alpha_{i_0}(t_{i_0})-1]}(A) = \{X_u = a_u,\ Z_v \in b_v;\ u \in U,\ v \in V\}$$

set

$$D = \{X_u = a_u,\ X_v = \Pi(b_v)\colon u \in U,\ v \in V\}$$

where $\Pi(k) = k = \Pi((2,k))$ $(k = 0, 1, 2)$.

Let

$$\tilde{B} = \pi_{J_{i_0}(s_{i_0})^c}(B) \cap D.$$

Finally for A, B as above set

$$A' = \pi_{[0,\alpha_{i_0}(t_{i_0})-1]}(A) = \pi_{[0,\alpha_{i_0}(t_{i_0})-1]}(\overline{B}).$$

LEMMA 4: *Using the notation developed and for $H \in \sigma(X_u\colon u \in I_{i_0}(t_{i_0}))$ and $G \in \sigma(X_u\colon u \in J_{i_0}(s_{i_0}))$ we have:*

(a) *If $G \cap \overline{B} \cap H \cap \overline{A} \neq \emptyset$ then*

$$(1-2^{-L})\,P(G)\,P(H)\,P(\tilde{B})\,P(\tilde{A}) \le P[G \cap \overline{B} \cap H \cap \overline{A}] \le P(G)\,P(H)\,P(\tilde{B})\,P(\tilde{A});$$

(b) $(1-2^{-i_0})\,P(\tilde{B}) \le P(\overline{B}) \le P(\tilde{B})$;

(c) $(1-2^{-L})\,P[A']\,P[\tilde{A}] \le P[A' \cap \tilde{A}] \le (1-2^{-L})^{-1}P[A']\,P[\tilde{A}]$;

(d) $(1-2^{-L})\,3^{-KN}\,3^{-K}\,P[A']\,P[\tilde{A}] \le P[A' \cap H \cap \overline{A}] \le (1-2^{-i_0})^{-1}3^{-KN}\,3^{-K}\,P[A']\,P[\tilde{A}]$;

(e) *Let M denote the number of cylinders in $B_0 \cap \overline{B}$ with coordinate set $[\alpha_L(s_L), -1]$; then*

$$(1-2^{-L})\,\frac{M}{N_{i_0}}\,P[\tilde{A}]\,P[\overline{B}] \le P[B_0 \cap \tilde{A} \cap \overline{B}] \le \frac{M}{N_{i_0}}\,P[\tilde{A}]\,P[\overline{B}].$$

Proof: (a) Let $E = \pi_{[\alpha_L(s_L),\omega_L(t_L)]^c}\,(C_L)$. E is measurable with respect to $\mathcal{F}_{-\infty}^{\alpha_L(s_L)-1} \vee \mathcal{F}_{\omega_L(t_L)+1}^{\infty}$ and $P(E) \ge P(C_L) \ge 1-2^{-L}$.

Let z be an atom of $\mathcal{F}_{-\infty}^{\alpha_L(s_L)-1} \vee \mathcal{F}_{\omega_L(t_L)_1}^{\infty}$, with $z \subseteq E$. By the construction of $\{Z_n\}$ and since $G \cap \overline{B} \cap H \cap \overline{A} \neq \emptyset$,

$$H \cap \overline{A} \cap G \cap \overline{B} \cap z = H \cap \tilde{A} \cap G \cap \tilde{B} \cap z$$

(that is, on $H \cap \tilde{A} \cap G \cap \tilde{B} \cap z$, $Z_{t_{i_0}-1} \neq 2$).

Thus $P(H \cap \overline{A} \cap G \cap \overline{B} \,|\, z) = P(H)\,P(\tilde{A})\,P(G)\,P(\tilde{B})$.

Integrate over $z \in E$:

$$\begin{aligned}
(1-2^{-L})\,&P(H)\,P(\tilde{A})\,P(G)\,P(\tilde{B}) \\
&\le P(E)\,P(H)\,P(\tilde{A})\,P(G)\,P(\tilde{B}) \\
&= \int_E P(H \cap \overline{A} \cap G \cap \overline{B} \,|z)\,P(dz) \\
&\le P(H \cap \overline{A} \cap G \cap \overline{B}) \\
&\le P(H)\,P(\tilde{A})\,P(G)\,P(\tilde{B}).
\end{aligned}$$

The last inequality follows because $H \cap \overline{A} \cap G \cap \overline{B} \neq \emptyset$, so $H \cap \overline{A} \cap G \cap \overline{B} \subseteq H \cap \tilde{A} \cap G \cap \tilde{B}$.

(b) Argue as in (a) with E replaced by $\pi_{[\alpha_L(s_L),\alpha_{i_0}(t_{i_0})-1]^c}(C_{i_0})$ and $\mathcal{F}_{-\infty}^{\alpha_L(s_L)-1}\vee\mathcal{F}_{\omega_L(t_L)+1}^{\infty}$ replaced by $\mathcal{F}_{-\infty}^{\alpha_L(s_L)-1}\vee\mathcal{F}_{\alpha_{i_0}(t_{i_0})}^{\infty}$.

(c) Use Lemma 2 (3) with l replaced by i_0 and the 0-coordinate replaced by $\alpha_{i_0}(t_{i_0})$.

(d) Let $A', \overline{A}, H$ be fixed.

Let $\{Q_j : j \in J'\}$ be the set of **X**-cylinders with coordinate set $[\alpha_L(s_L), -K_{i_0}-1]$ on some backward frame. Let Q_j^0 be the unique **X**-cylinder with coordinate set $[\alpha_L(s_L), -1]$ so

$$Q_j \cap A' \cap \overline{A} \cap H = Q_j^0 \cap A' \cap \overline{A} \cap H \neq \emptyset.$$

Also $A' \cap Q_j^0 = G \cap \overline{B}_j$ for some unique $G \in \mathcal{F}_{s_{i_0}+1}^{s_{i_0}+\log N_{i_0}}$ and $B_j = Q_j^0 \cap A'$. Then by (a), and since the union over all Q_j has measure one,

$$\begin{aligned}
P(A' \cap H \cap \overline{A}) &= \sum_j P[A' \cap H \cap \overline{A} \cap Q_j] \\
&= \sum_j P[A' \cap H \cap \overline{A} \cap Q_j^0] \\
&= \sum_j P[\overline{A} \cap G \cap \overline{B_j} \cap H] \\
&\geq (1-2^{-L}) \sum_j P(G)P(\tilde{A})P(H)P(\tilde{B}_j) \\
&= (1-2^{-L})3^{-NK} \sum_j P(\tilde{A})P(A')P(Q_j^0) \\
&= (1-2^{-L})3^{-NK}3^{-K}\, P(\tilde{A})P(A') \sum_j P(Q_j) \\
&= (1-2^{-L})3^{-NK}3^{-K} P(\tilde{A})P(A').
\end{aligned}$$

The upper bound is similar.

(e) Repeat the argument for part (a) with $\tilde{A}$ replacing $\overline{A}$ and the entire probability space replacing H. ∎

KEY LEMMA: *Given S, B_0, $f = \sum f_A 1_A \in \mathcal{L}(i_0)$ as above, define $g \in \mathcal{L}(i_0 - 1)$ by the conditional expectation formula*

$$g = \sum_{A' \cap \tilde{A}} \left(P[A' \cap \tilde{A}]^{-1} \sum_{A \subseteq A' \cap \tilde{A}} f_A\, P[A] \right) 1_{A' \cap \tilde{A}}.$$

Then for any $\delta > 0$ we have

$$\left| \int_{B_0} f\,dP - \int_{B_0} g\,dP \right| \le \left(\frac{4 \cdot 3^{K_{i_0}}}{N_{i_0}\delta^2} + 4\delta + 20 \cdot 2^{-i_0} \right) P[S]$$

and

$$\left| \int f\,dP - \int g\,dP \right| \le \left(\frac{4 \cdot 3^{K_{i_0}}}{N_{i_0}\delta^2} + 4\delta + 20 \cdot 2^{-i_0} \right) P[S^+].$$

Proof: During the proof use K for K_{i_0} and N for N_{i_0}. By definition

$$\int_{B_0} f\,dP = \sum_{\overline{A},\overline{B}} \int_{B_0 \cap \overline{A} \cap \overline{B}} f\,dP,$$

where the summation extends over all $\overline{A}$ which can be obtained from A's in the representation of f and over all $\overline{B}$ which can be obtained for a fixed $\overline{A}$ from a set $B \subset B_0$.

Fix such an $\overline{A}$ and $\overline{B}$. Define $\mathcal{G}(\overline{B}) = \{G\colon G = \{X_u = c_u\colon u \in J_{i_0}(s_{i_0})\}\}$ so $\overline{B} = \bigcup_{G \in G(\overline{B})} G \cap \overline{B}$. Also on each cylinder $C \subseteq B \cap \overline{B}$ the random choice $q(C) = q(x,\, s_{i_0})$ is well-defined giving us a subset

$$M(\overline{B}) = M = \{q(C)\colon C \subseteq B \cap \overline{B}\} \subseteq \{0, 1, \ldots, N-1\}.$$

To complete our notational scaffolding define

$$\mathcal{H}(\overline{A}) = \{H\colon H = \{X_u = c_u\colon u \in I_{i_0}(t_{i_0})\}\},$$

so

$$\overline{A} = \bigcup_{H \in \mathcal{H}(\overline{A})} H \cap \overline{A}.$$

Now

$$\begin{aligned}
\int_{B_0 \cap \overline{A} \cap \overline{B}} f\,dP &= \sum_{\substack{A:\\ A \subseteq \overline{A}}} f_A\, P[B_0 \cap \overline{A} \cap \overline{B} \cap A] \\
&= \sum_{H \in \mathcal{H}(\overline{A})} \sum_{\substack{G \in \mathcal{G}(\overline{B}):\\ G \cap \overline{B} \subseteq B_0 \cap \overline{B}\\ H \cap \overline{A} \cap G \cap \overline{B} \neq \emptyset}} f_{\overline{A} \cap H \cap A'}\, P[H \cap \overline{A} \cap G \cap \overline{B}] \\
&\le \sum_{H \in \mathcal{H}(\overline{A})} \sum_{\substack{G \in \mathcal{G}(\overline{B}):\\ G \cap \overline{B} \subseteq B_0 \cap \overline{B}\\ H \cap \overline{A} \cap G \cap \overline{B} \neq \emptyset}} f_{\overline{A} \cap H \cap A'}\, P[H]\, P[G]\, P[\tilde{A}]\, P[\tilde{B}] \qquad (3.1)
\end{aligned}$$

by Lemma 4(a).

Now we use Chebychev's Inequality to estimate the number of terms in the second sum so that $H \cap \overline{A} \cap G \cap \overline{B} \neq \emptyset$.

Parenthetically let $\Omega = \{0,1,2\}^K$ and fix $\omega_0 \in \Omega$. By Chebychev's Inequality the number of words $\omega = (\omega_1, \ldots, \omega_N) \in \Omega^N$ satisfying

$$(1-\delta)\,|M|\,3^{-K} \leq |\{k \in M: \omega_{k+1} = \omega_0\}| \leq (1+\delta)\,|M|\,3^{-K}$$

is at least $(1 - \frac{3^K}{\delta^2 |M|})\,3^{KN}$ where $|\cdot|$ denotes cardinality and

$$M \subseteq \{0, \ldots, N-1\}.$$

This is seen in the usual way: Choose $\omega \in \Omega^N$ uniformly, i.e. with probability 3^{-KN}.

Let $W(\omega) = |\{k \in M: \omega_{k+1} = \omega_0\}|$, then $E[W] = |M|\,3^{-K}$ and $\mathrm{Var}\,[W] = |M|\,3^{-K}\,(1-3^{-K})$. Hence

$$\mathrm{Prob}\,\big[\,|W - E[W]| < |M|\,3^{-K}\delta\big] \geq 1 - \frac{|M|\,3^{-K}\,(1-3^{-K})}{|M|^2\,3^{-2K}\,\delta^2} \geq 1 - \frac{3^K}{|M|\delta^2},$$

so the number of such ω is $\geq (1 - \frac{3^K}{|M|\delta^2})\,3^{KN}$.

Now a word $\omega \in \Omega^N$ determines a cylinder $H \in \mathcal{H}(\overline{A})$ canonically (recall that H is $\sigma(X_u: u \in I_{i_0}(t_{i_0}))$-measurable and $|I_{i_0}(t_{i_0})| = KN$). The preceding calculation shows that for at least

$$\left(1 - \frac{3^K}{\delta^2 |M(\overline{B})|}\right) 3^{KN} \qquad \text{cylinders} \quad H \in \mathcal{H}(\overline{A})$$

we have

$$\begin{aligned}(1-\delta)\,|M(\overline{B})|\,3^{-K} &\leq |\{G \in \mathcal{G}(\overline{B}): G \cap \overline{B} \subseteq B_0 \cap \overline{B},\ G \cap \overline{B} \cap H \cap \overline{A} \neq \emptyset\}| \\ &\leq (1+\delta)\,|M(\overline{B})|\,3^{-K}.\end{aligned}$$

Let $F_0 = \{H \in \mathcal{H}(\overline{A})$ with this property$\}$.

Continuing with (3.1) and using Lemma 4(b), we get

$$
\begin{aligned}
&\int_{B_0\cap\overline{A}\cap\overline{B}} f\,dP \\
&\le \sum_{H\in\mathcal{H}(\overline{A})} \sum_{\substack{G\in\mathcal{G}(\overline{B}):\\ G\cap\overline{B}\subseteq B_0\cap\overline{B}\\ H\cap\overline{A}\cap G\cap\overline{B}\neq\emptyset}} f_{\overline{A}\cap H\cap A'}\, P[H]\,P[G]\,P[\tilde{A}]\,P[\tilde{B}] \\
&\le 3^{-KN} N^{-1} \frac{(1+\delta)}{(1-2^{-i_0})} 3^{-K}\, |M(\overline{B})| \sum_{H\in F_0} f_{\overline{A}\cap H\cap A'} \quad P[\tilde{A}]\,P[\overline{B}] \\
&\quad + (1-2^{-i_0})^{-1}\, 3^{-KN} N^{-1} 3^{-K} \frac{3^K\, 3^{KN}}{\delta^2|M(\overline{B})|} \sum_{\substack{G\in\mathcal{G}(\overline{B})\\ G\cap\overline{B}\subseteq B_0\cap\overline{B}}} P[\tilde{A}]\,P[\overline{B}].
\end{aligned}
$$

(We use the fact that $0 \le f_A \le 1$.)

The lower bound is similar, using Lemma 4 again:

$$
\begin{aligned}
&\int_{B_0\cap\overline{A}\cap\overline{B}} f\,dP \ge \\
&(1-2^{-L})\, 3^{-KN} N^{-1} (1-\delta)\, 3^{-K}\, |M(\overline{B})| \sum_{H\in F_0} f_{\overline{A}\cap H\cap A'}\, P[\tilde{A}]\,P[\overline{B}].
\end{aligned}
$$

Thus

$$
\int_{B_0\cap\overline{A}\cap\overline{B}} f\,dP = 3^{-KN} N^{-1} 3^{-K}\, |M(\overline{B})| \sum_{H\in\mathcal{H}(\overline{A})} f_{\overline{A}\cap H\cap A'}\, P[\tilde{A}]\,P[\overline{B}] + \alpha_1\, 3^{-K}
$$

where $|\alpha_1| \le \left(\frac{2\cdot 3^K}{N\delta^2} + 2\delta + 2\,2^{-i_0}\right) P[\tilde{A}]\,P[\overline{B}]$.

Now sum over all $A \subseteq A' \cap \tilde{A}$ and get

$$
(3.2)\quad \int_{B_0\cap\tilde{A}\cap\overline{B}} f\,dP = \left(\sum_{A\subseteq A'\cap\tilde{A}} f_A\right) 3^{-KN} N^{-1} 3^{-K}\, |M(\overline{B})|\, P[\tilde{A}]\,P[\overline{B}] + \alpha_1
$$

since the number of such A is 3^K.

On the other hand, using Lemma 4(b)–(e)

$$\begin{aligned}
&\int_{B_0\cap\tilde{A}\cap\overline{B}} g\,dP \\
&= \int_{B_0\cap\tilde{A}\cap\overline{B}} P[A'\cap\tilde{A}]^{-1} \sum_{A\subseteq A'\cap\tilde{A}} f_A P[A]\, 1_{A'\cap\tilde{A}}\, dP \\
&= P[A'\cap\tilde{A}]^{-1} \sum_{A\subseteq A'\cap\tilde{A}} f_A\, P[A]\, P[B_0\cap\tilde{A}\cap\overline{B}] \\
&\le \frac{(1-2^{-L})^{-1}}{P[A']P[\tilde{A}]} \sum_{\substack{A\subseteq A'\cap\tilde{A}\\ A=A'\cap\overline{A}\cap H}} f_A\, P[A'\cap\overline{A}\cap H]\, \frac{|M(\overline{B})|}{N}\, P[\tilde{A}]\, P[\overline{B}] \\
&\le (1-2^{-i_0})^{-1}\frac{(1-2^{-L})^{-1}}{P[A']P[\tilde{A}]} \sum_{A\subseteq A'\cap\tilde{A}} f_A\, 3^{-KN}3^{-K}\, P[A']\, P[\tilde{A}]\, \frac{|M(\overline{B})|}{N}\, P[\tilde{A}]\, P[\overline{B}] \\
&\le (1-2^{-i_0})^{-2}\, 3^{-KN}3^{-K} N^{-1}\, |M(\overline{B})|\, P[\tilde{A}]\, P[\overline{B}] \left(\sum_{A\subseteq A'\cap\tilde{A}} f_A\right).
\end{aligned}$$

The lower bound is similar:

$$\int_{B_0\cap\tilde{A}\cap\overline{B}} g\,dP \quad \ge (1-2^{-L})^2\, 3^{-K}3^{-KN} N^{-1}\, |M(\overline{B})|\, P[\tilde{A}]\, P[\overline{B}] \left(\sum_{A\subseteq A'\cap\tilde{A}} f_A\right).$$

So

$$\int_{B_0\cap\tilde{A}\cap\overline{B}} g\,dP = 3^{-K}3^{-KN} N^{-1}\, |M(\overline{B})|\, P[\tilde{A}]\, P[\overline{B}] \left(\sum_{A\subseteq A'\cap\tilde{A}} f_A\right) + \alpha_2$$

where $\alpha_2 \le (8\cdot 2^{-i_0})P[\tilde{A}]\,P[\overline{B}]$. (Note that there are at most $3^{(K+1)N}$ sets $A\subset A'\cap\tilde{A}$.)

So summing over $\tilde{A}$, $\overline{B}$ gives

$$\begin{aligned}
\left|\int_{B_0} f\,dP - \int_{B_0} g\,dP\right| &\le \sum_{\tilde{A},\overline{B}} \left|\int_{B_0\cap\tilde{A}\cap\overline{B}} f\,dP - \int_{B_0\cap\tilde{A}\cap\overline{B}} g\,dP\right| \\
&\le \sum_{\tilde{A},\overline{B}} P[\tilde{A}]\,P[\overline{B}] \left(\frac{2\cdot 3^K}{N\delta^2} + 2\delta + 10\cdot 2^{-i_0}\right) \\
&= \left(\frac{4\cdot 3^K}{N\delta^2} + 4\delta + 20\cdot 2^{i_0}\right) P[S^+]\,P[S^-] \\
&= \left(\frac{4\cdot 3^K}{N\delta^2} + 4\delta + 20\cdot 2^{-i_0}\right) P[S].
\end{aligned}$$

Taking $B_0 = S^-$ and summing over all backward frames gives

$$\left| \int f\,dP - \int g\,dP \right| \le \left(\frac{4 \cdot 3^K}{N\delta^2} + 4\delta + 20 \cdot 2^{-i_0} \right) P[S^+]. \qquad \blacksquare$$

LEMMA 5: *For each $\varepsilon > 0$ there is an l so $\forall\, L > l$ and S, B_0, $f = \sum f_A 1_A \in \mathcal{L}(L)$ as above we have*

$$\left| \int_{B_0} f\,dP - P[B_0] \int f\,dP \right| < \varepsilon P[S].$$

Proof: Using the Key Lemma $L - l + 1$ times we find a function $h \in \mathcal{L}(l-1)$, so $h \in \mathcal{F}_0^\infty$ and

$$\left| \int_{B_0} f\,dP - \int_{B_0} h\,dP \right| \le \sum_{i=l}^{\infty} \left(20 \cdot 2^{-i} + 4\delta_i + \frac{4 \cdot 3^{K_i}}{\delta_i^2 N_i} \right) P[S],$$

$$\left| P[B_0] \int f\,dP - P[B_0] \int h\,dP \right| \le \sum_{i=l}^{\infty} \left(20 \cdot 2^{-i} + 4\delta_i + \frac{4 \cdot 3^{K_i}}{\delta_i^2 N_i} \right) P[S],$$

since $B_0 \subseteq S^-$, $P[B_0]\,P[S^+] \le P[S^-]\,P[S^+] = P[S]$.

Also since $B_0 \in \mathcal{F}_{-\infty}^{-1}$, $h \in \mathcal{F}_0^\infty$ they are independent, so

$$\int_{B_0} h\,dP = P[B_0] \int h\,dP.$$

The above sum is the tail of a convergent series, so l may be taken large enough that the sum is less than $\varepsilon/2$. $\blacksquare$

Proof of Theorem 2: We show that, given ε, there is an n such that if $B_1 \in \mathcal{G}_{-\infty}^{-n}$ and f is $\mathcal{G}_n^\infty$-measurable, $0 \le f \le 1$, then

$$\left| \int_{B_1} f\,dP - P[B_1] \int f\,dP \right| < \varepsilon.$$

Fix $\varepsilon' > 0$. Find l so $P[C_l] > 1 - \varepsilon'$ and choose $n = 3(\Lambda_l + \beta_l)$. Also, assume l is large enough that the preceding lemma holds with ε' for ε.

Note that

$$\left| \int_{B_1} f\,dP - P(B_1) \int f\,dP \right| \le \left| \int_{B_1 \cap C_l} f\,dP - P(B_1 \cap C_l) \int_{C_l} f\,dP \right| + 3\varepsilon',$$

and hence it suffices to show that for $B_2 \in \mathcal{G}_{-\infty}^{-n} \cap C_l$ and f which is $\mathcal{G}_n^{\infty} \cap C_l$-measurable,

$$\left| \int_{B_2} f\,dP - P(B_2) \int f\,dP \right| < \varepsilon'. \tag{3.3}$$

Since $n \geq \max\{\beta_l, \Lambda_l\}$, by Lemma 2 it suffices to show (3.3) for $B \in \mathcal{F}_{-\infty}^{-\beta_l+\Lambda_l} \cap C_l$ and f $\mathcal{H}_l \cap C_l$ measurable. Finally, we see that it suffices to show (3.3) for B_3 a finite union of $\mathbf{X}$-cylinders in $\mathcal{F}_{-\infty}^{-\beta_l+\Lambda_l} \cap C_l$ and for $f_1 = \sum_{A \text{ disjoint}} f_A 1_A$ with $0 \leq f_A \leq 1$ and A a cylinder set in $\mathcal{H}_l$ so that $A \cap C_l \neq \emptyset$. Since the frames S with indices l and L form a disjoint partition of C_l, it suffices therefore to show that for ε' sufficiently small

$$\left| \int_{B_3 \cap S^-} 1_{S^+} f_1\,dP - P(B_3 \cap S^-) \int_{S^+} f_1\,dP \right| \leq \varepsilon' P(S).$$

But this statement has been shown in Lemma 5, for l large enough with $B_0 = B_3 \cap S^-$ and $f = 1_{S^+} f_1$. ∎

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